

SINGULAR LOCI OF HIBI TORIC VARIETIES

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ABSTRACT. We first construct explicit bases for the cotangent spaces at singular points on Hibi toric varieties, i.e., toric varieties associated to finite distributive lattices. We then determine the singular loci of these toric varieties.

INTRODUCTION

Let K denote the base field which we assume to be algebraically closed of arbitrary characteristic. Given a distributive lattice \mathcal{L} , let $X_{\mathcal{L}}$ denote the *Hibi variety* - the affine variety in $\mathbb{A}^{\#\mathcal{L}}$ whose vanishing ideal is generated by the binomials $X_{\tau}X_{\varphi} - X_{\tau \vee \varphi}X_{\tau \wedge \varphi}$ in the polynomial algebra $K[X_{\alpha}, \alpha \in \mathcal{L}]$; here, $\tau \vee \varphi$ (resp. $\tau \wedge \varphi$) denotes the *join* - the smallest element of \mathcal{L} greater than both τ, φ (resp. the *meet* - the largest element of \mathcal{L} smaller than both τ, φ). In the sequel, we shall refer to the quadruple $(\tau, \varphi, \tau \vee \varphi, \tau \wedge \varphi)$ ((τ, φ) being a skew pair) as a *diamond*. These varieties were extensively studied by Hibi in [16] where Hibi proves that $X_{\mathcal{L}}$ is a normal variety. On the other hand, Eisenbud-Sturmfels show in [10] that a binomial prime ideal is toric (here, “toric ideal” is in the sense of [27]). Thus one obtains that $X_{\mathcal{L}}$ is a normal toric variety. We refer to such an $X_{\mathcal{L}}$ as a Hibi toric variety.

For \mathcal{L} being the Bruhat poset of Schubert varieties in a minuscule G/P , it is shown in [12] that $X_{\mathcal{L}}$ flatly deforms to G/P , i.e., there exists a flat family over \mathbb{A}^1 with G/P as the generic fiber and $X_{\mathcal{L}}$ as the special fiber. More generally for a Schubert variety $X(w)$ in a minuscule G/P , it is shown in [12] that $X_{\mathcal{L}_w}$ flatly deforms to $X(w)$ (here, \mathcal{L}_w is the Bruhat poset of Schubert subvarieties of $X(w)$). In a subsequent paper (cf. [14]), the authors of loc.cit., studied the singularities of $X_{\mathcal{L}}$, \mathcal{L} being the Bruhat poset of Schubert varieties in the Grassmannian; further, in loc.cit., the authors gave a conjecture giving a necessary and sufficient condition for a point on $X_{\mathcal{L}}$ to be smooth, and proved in loc.cit., the sufficiency part of the conjecture. Subsequently, the necessary part of the conjecture was proved in [3] by Batyrev et al. The toric varieties $X_{\mathcal{L}}$, \mathcal{L} being the Bruhat poset of Schubert varieties in the Grassmannian play an important role in the area of mirror symmetry; for more details, see [2, 3]. We refer to such an $X_{\mathcal{L}}$ as a *Grassmann-Hibi* toric variety.

In this paper, we first determine an explicit description (cf. §4) of the cone σ associated to a Hibi toric variety and the dual cone σ^{\vee} (note that if S_{σ} denotes the

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semigroup of integral points in σ^\vee , then $K[X_\mathcal{L}]$, the K -algebra of regular functions on $X_\mathcal{L}$ can be identified with the semigroup algebra $K[S_\sigma]$. Using this we determine the cotangent space at any point on $X_\mathcal{L}$ very explicitly as described below.

For each face τ of σ , there exists a distinguished point P_τ in the torus orbit \mathcal{O}_τ corresponding to τ ; namely, identifying a (closed) point in $X_\mathcal{L}$ with a semigroup map $S_\sigma \rightarrow K^* \cup \{0\}$, P_τ corresponds to the semigroup map f_τ which sends $u \in S_\sigma$ to 1 or 0 according as u is in τ^\perp or not. We have that f_τ induces an algebra map $\psi_\tau : K[X_\mathcal{L}] \rightarrow K$ whose kernel is precisely M_{P_τ} , the maximal ideal corresponding to P_τ . For $\alpha \in \mathcal{L}$, we shall denote by $P_\tau(\alpha)$, the co-ordinate of P_τ (considered as a point of $\mathbb{A}^{\#\mathcal{L}}$) corresponding to α . Set

$$D_\tau = \{\alpha \in \mathcal{L} \mid P_\tau(\alpha) \neq 0\}$$

It turns out that D_τ is an embedded sublattice of \mathcal{L} ; further, D_τ determines the local behavior at P_τ . To make this more precise, identifying $K[X_\mathcal{L}]$ as the quotient of the polynomial algebra $K[X_\alpha, \alpha \in \mathcal{L}]$ by the ideal generated by the binomials $X_\tau X_\varphi - X_{\tau \vee \varphi} X_{\tau \wedge \varphi}$, let x_α denote the image of X_α in $K[X_\mathcal{L}]$. For $\alpha \in \mathcal{L}$, set

$$F_\alpha = \begin{cases} x_\alpha, & \text{if } \alpha \notin D_\tau \\ 1 - x_\alpha, & \text{if } \alpha \in D_\tau \end{cases}$$

Then we have that M_{P_τ} is generated by $\{F_\alpha, \alpha \in \mathcal{L}\}$. Fix a maximal chain Γ in D_τ . Let $\Lambda_\tau(\Gamma)$ denote the sublattice of \mathcal{L} consisting of all maximal chains of \mathcal{L} containing Γ . Let E_τ denote the set of all $\alpha \in \mathcal{L}$ such that there exists a $\beta \in D_\tau$ such that (α, β) is a diagonal of a diamond whose other diagonal is contained in $\mathcal{L} \setminus D_\tau$. Let $Y_\tau(\Gamma) = \Lambda_\tau(\Gamma) \setminus E_\tau$. For $F \in M_{P_\tau}$, let \overline{F} denote the class of F in $M_{P_\tau} / M_{P_\tau}^2$. We define an equivalence relation (cf. §5.4.2) on $\mathcal{L} \setminus D_\tau$ in such a way that for two elements θ, θ' in the same equivalence class, we have $\overline{F}_\theta = \overline{F}_{\theta'}$. Denoting by $G_\tau(\Gamma)$ the set of equivalence classes $[\theta]$, where $\theta \in Y_\tau(\Gamma) \setminus \{\Gamma \cup E_\tau\}$ and $\overline{F}_{[\theta]}$ is non-zero in $M_{P_\tau} / M_{P_\tau}^2$, we have

Theorem 1: (cf. Theorem 6.7) $\{\overline{F}_{[\theta]}, [\theta] \in G_\tau(\Gamma)\} \cup \{\overline{F}_\gamma, \gamma \in \Gamma\}$ is a basis for the cotangent space $M_{P_\tau} / M_{P_\tau}^2$.

Using the above result, we determine $\text{Sing } X_\mathcal{L}$ (cf. Theorem 6.9):

Let $\mathcal{S}_\mathcal{L} = \{\tau < \sigma \mid \#G_\tau(\Gamma) + \#\Gamma > \#\mathcal{L}\}$.

Theorem 2: (cf. Theorem 6.9) $\text{Sing } X_\mathcal{L} = \bigcup_{\tau \in \mathcal{S}_\mathcal{L}} X(D_\tau)$.

(Here, $X(D_\tau)$ denotes the Hibi variety associated to the distributive lattice D_τ .)

Sketch of the proof of the main theorem: Using our explicit description of the generators for σ, σ^\vee (cf. §4), we first determine explicitly the embedded sublattice associated to a face τ of σ . We then analyze the local expression around P_τ for any $f \in I(X_\mathcal{L})$, the vanishing ideal of $X_\mathcal{L}$, and show the generation of the degree one part of $\text{gr}(R_\mathcal{L}, M_{P_\tau})$ by $\{\overline{F}_\theta, \theta \in Y_\tau(\Gamma)\}$ (here, $R_\mathcal{L} = K[X_\mathcal{L}]$, and M_{P_τ} is the maximal ideal

in $R_{\mathcal{L}}$ corresponding to P_{τ}). We then define the equivalence relation on $\mathcal{L} \setminus D_{\tau}$. The linear independence of $\{\overline{F}_{[\theta]} [\theta] \in G_{\tau}(\Gamma)\} \cup \{\overline{F}_{\gamma}, \gamma \in \Gamma\}$ in $M_{P_{\tau}} / M_{P_{\tau}}^2$ is proved using the defining equations of $X_{\mathcal{L}}$ (as a closed subvariety of $\mathbb{A}^{\#\mathcal{L}}$), thus proving Theorem 1. Theorem 2 is then deduced from Theorem 1.

The sections are organized as follows: In §1, we recall generalities on toric varieties. In §2, we recall some basic results on distributive lattices. In §3, we introduce the Hibi toric variety $X_{\mathcal{L}}$, and recall some basic results on $X_{\mathcal{L}}$. In §4, we determine generators for the cone σ (and the dual cone σ^{\vee}); further, for a face τ of σ , we introduce D_{τ} and derive some properties of D_{τ} . In §5, we prove the generation of $M_{P_{\tau}} / M_{P_{\tau}}^2$ by $\{\overline{F}_{[\theta]}, [\theta] \in G_{\tau}(\Gamma)\} \cup \{\overline{F}_{\gamma}, \gamma \in \Gamma\}$, while the linear independence is proved in §6.

1. GENERALITIES ON TORIC VARIETIES

Since our main object of study is a certain affine toric variety, we recall in this section some basic definitions on affine toric varieties. Let $T = (K^*)^m$ be an m -dimensional torus.

Definition 1.1. (cf. [11], [18]) An *equivariant affine embedding* of a torus T (or also *equivariant affine toroidal embedding*) is an affine variety $X \subseteq \mathbb{A}^l$ containing T as a dense open subset and equipped with a T -action $T \times X \rightarrow X$ extending the action $T \times T \rightarrow T$ given by multiplication. If in addition X is normal, then X is called an *affine toric variety*.

1.2. Résumé of combinatorics of affine toric varieties. Let M be the character group of T , and N the \mathbb{Z} -dual of M . Let X be an equivariant affine toroidal embedding of T ; let $R = K[X]$ (the ring of regular functions on X). We note the following:

- **M -gradation:** We have an action of T on R , and hence writing R as a sum of T weight spaces, we obtain a M -grading for R : $R = \bigoplus_{\chi \in M} R_{\chi}$, where

$$R_{\chi} = \{f \in R \mid tf = \chi(t)f, \forall t \in T\}$$

- **The semi group S :** Let $S = \{\chi \in M \mid R_{\chi} \neq 0\}$. Then via the multiplication in R , S acquires a semi group structure. Thus S is a sub semigroup of M , and R gets identified with the semi group algebra $K[S]$.

- **Finite generation of S :** In view of the fact that R is a finitely generated K -algebra, we obtain that S is a finitely generated sub semigroup of M .

- **Generation of M by S :** The fact that T and X have the same function field (since T is a dense open subset of X) implies that M is generated by S (note that $K[T]$ is the group algebra $K[M]$).

Thus given an equivariant affine toroidal embedding X (of the torus T), X determines a finitely generated, sub semigroup S of M which generates M in such a way that the semi group algebra $K[S]$ is the ring of regular functions on X . Conversely, given a finitely generated, sub semigroup S of M which generates M , we have clearly that $X := \text{Spec } K[S]$ is an equivariant affine toroidal embedding of T .

Thus we obtain a bijection between $\{\text{equivariant affine toroidal embeddings of } T\}$ and $\{\text{finitely generated, sub semigroups of } M \text{ which generate } M\}$.

Let notation be as above.

• **Saturation of S :** We have (see [18] for example) that the ring R (being identified as the semi group algebra $K[S]$) is normal if and only if S is saturated (recall that a sub semigroup A in M is *saturated*, if for $\chi \in M$, $r\chi \in A \implies \chi \in A$ (where r is an integer > 1 ; equivalently, A is precisely the lattice points in the cone generated by A , i.e., $A = \theta \cap A$ where $\theta = \sum a_i x_i, a_i \in \mathbb{R}_+, x_i \in A$)).

Thus given an affine toric variety X (with torus action by T), X determines a finitely generated, saturated sub semigroup S of M which generates M in such a way that the semi group algebra $K[S]$ is the ring of regular functions on X .

Let S be as above, namely, a finitely generated, saturated, sub semigroup of M which generates M . Let $\theta := \sum a_i x_i, a_i \in \mathbb{R}_+, x_i \in S$ be the cone generated by S . Then θ is a *rational convex polyhedral cone* (i.e., a cone with a (finite) set of lattice points as generators); further, θ is not contained in any hyperplane in $M_{\mathbb{R}} := M \otimes \mathbb{R}$ (since S generates M , we have that θ generates $M \otimes \mathbb{R}$). Let

$$\sigma := \theta^{\vee} = \{v \in N_{\mathbb{R}} | v(f) \geq 0, \forall f \in \theta\}$$

(note that $N_{\mathbb{R}} (= N \otimes \mathbb{R})$ is the linear dual of $M_{\mathbb{R}}$). Then σ is a *strongly convex rational polyhedral cone* (one may take this to be the definition of a strongly convex rational convex polyhedral cone, namely, a rational convex polyhedral cone τ is *strongly convex* if τ^{\vee} is not contained in any hyperplane; equivalently, τ does not contain any non-zero linear subspace.)

Thus an affine toric variety X (with a torus action by T) determines a strongly convex, rational polyhedral cone σ (inside $N_{\mathbb{R}} (= M_{\mathbb{R}}^*)$) in such a way that $K[X]$ is the semigroup algebra $K[S_{\sigma}]$, where S_{σ} is the sub semigroup in M consisting of the set of lattice points in σ^{\vee} .

Conversely, starting with a strongly convex rational polyhedral cone σ , $S_{\sigma} := \sigma^{\vee} \cap M$ is a finitely generated, saturated, sub semigroup of M which generates M ; and hence determines an affine toric variety X (with a torus action by T), namely, $X = \text{Spec } K[S_{\sigma}]$.

This sets up a bijection between $\{\text{affine toric varieties with torus action by } T\}$, $\{\text{finitely generated, saturated, sub semigroups of } M \text{ which generate } M\}$, and $\{\text{strongly convex rational polyhedral cones inside } N_{\mathbb{R}}\}$.

In the sequel, we shall denote $\text{Spec } K[S_{\sigma}]$ by X_{σ} .

1.3. Toric ideals, binomial prime ideals and equivariant affine toroidal embeddings: In this subsection we shall establish bijections between $\{\text{equivariant affine toroidal embeddings}\}$, $\{\text{toric ideals}\}$ and $\{\text{binomial prime ideals}\}$ (here, by a *binomial ideal* we mean an ideal generated by binomials, i.e., polynomials with at most two terms).

Let $\mathcal{A} = \{\chi_1, \dots, \chi_l\}$ be a subset of \mathbb{Z}^d . Consider the map

$$\pi_{\mathcal{A}} : \mathbb{Z}_+^l \rightarrow \mathbb{Z}^d, \quad \mathbf{u} = (u_1, \dots, u_l) \mapsto u_1\chi_1 + \dots + u_l\chi_l.$$

Let $K[\mathbf{x}] := K[x_1, \dots, x_l]$, $K[\mathbf{t}^{\pm 1}] := K[t_1, \dots, t_d, t_1^{-1}, \dots, t_d^{-1}]$.

The map $\pi_{\mathcal{A}}$ induces a homomorphism of semigroup algebras

$$\widehat{\pi}_{\mathcal{A}} : K[\mathbf{x}] \rightarrow K[\mathbf{t}^{\pm 1}], \quad x_i \mapsto \mathbf{t}^{\chi_i}.$$

(If $\chi_i = (a_{i1}, \dots, a_{id})$, then $\mathbf{t}^{\chi_i} = t_1^{a_{i1}} \dots t_d^{a_{id}}$.)

Definition 1.4. (cf. [27]) The kernel of $\widehat{\pi}_{\mathcal{A}}$ is denoted by $I_{\mathcal{A}}$ and called the *toric ideal* associated to \mathcal{A} .

Note that a toric ideal is prime. We shall now show that we have a bijection between $\{\text{toric ideals}\}$ and $\{\text{equivariant affine toroidal embeddings}\}$

Consider the action of $T(= (K^*)^d)$ on \mathbb{A}^l given by

$$\mathbf{t}(a_1, \dots, a_l) = (\mathbf{t}^{\chi_1} a_1, \dots, \mathbf{t}^{\chi_l} a_l)$$

Then $\mathcal{V}(I_{\mathcal{A}})$, the affine variety of the zeroes in K^l of $I_{\mathcal{A}}$, is simply the Zariski closure of the T -orbit through $(1, 1, \dots, 1)$. Let $M_{\mathcal{A}}$ be the \mathbb{Z} -span of \mathcal{A} (inside $X(T) = \mathbb{Z}^d$), and $d_{\mathcal{A}}$ the rank of $M_{\mathcal{A}}$. Let $S_{\mathcal{A}}$ be the sub semigroup of \mathbb{Z}^d generated by \mathcal{A} . Let $T_{\mathcal{A}}$ be the $d_{\mathcal{A}}$ -dimensional torus with $M_{\mathcal{A}}$ as the character group. Then we have

Proposition 1.5. $\mathcal{V}(I_{\mathcal{A}})$ is an equivariant affine embedding of $T_{\mathcal{A}}$ (of dimension $d_{\mathcal{A}}$); further, $K[\mathcal{V}(I_{\mathcal{A}})]$ is the semigroup algebra $K[S_{\mathcal{A}}]$.

Remark 1.6. In the above definition, we do not require $\mathcal{V}(I_{\mathcal{A}})$ to be normal. Note that $\mathcal{V}(I_{\mathcal{A}})$ is normal if and only if $S_{\mathcal{A}}$ is saturated. A variety of the form $\mathcal{V}(I_{\mathcal{A}})$, is also sometimes called an *affine toric variety*. But in this paper, by an affine variety, we shall mean a normal toric variety as defined in Definition 1.1.

1.6.1. *The vanishing ideal of an equivariant affine toroidal embedding:* Conversely, let X be an equivariant affine toroidal embedding (with a torus action by $T(= (K^*)^d)$ as in Definition 1.1). Let $K[X] = K[S]$, for some suitable finitely generated sub semigroup of $X(T) (= \mathbb{Z}^d)$. Fix a set of generators $\mathcal{A} := \{\chi_1, \dots, \chi_l\}$ for the sub semigroup S of \mathbb{Z}^d . We have a surjection

$$K[x_1, \dots, x_l] \rightarrow K[X], \quad x_i \mapsto \mathbf{t}^{\chi_i}$$

whose kernel equals $I_{\mathcal{A}}$. Thus we obtain

Proposition 1.7. With X as above, we have that $I(X)$, the vanishing ideal of X (for the embedding $X \hookrightarrow \mathbb{A}^l$) is a toric ideal.

Thus we obtain a bijection between $\{\text{equivariant, affine toroidal embeddings}\}$ and $\{\text{toric ideals}\}$. Next we want to show that we have a bijection between $\{\text{equivariant, affine toroidal embeddings}\}$ and $\{\text{binomial prime ideals}\}$.

First, we recall the following (see [27]).

Proposition 1.8. *The toric ideal $I_{\mathcal{A}}$ is spanned as a K -vector space by the set of binomials*

$$\{\mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}} \mid \mathbf{u}, \mathbf{v} \in \mathbb{Z}_+^l \text{ with } \pi_{\mathcal{A}}(\mathbf{u}) = \pi_{\mathcal{A}}(\mathbf{v})\}. \quad (*)$$

As an immediate consequence, we obtain

Corollary 1.9. *A toric ideal is a binomial prime ideal.*

In view of the above bijection, we have the following:

Reformulation of Corollary 1.9: The vanishing ideal of an equivariant, affine toroidal embedding is binomial prime.

Conversely, we shall now show that a binomial prime ideal defines an equivariant affine toroidal embedding.

1.10. Varieties defined by binomials. Let $l \geq 1$, and let X be an affine variety in \mathbb{A}^l , not contained in any of the coordinate hyperplanes $\{x_i = 0\}$. Further, let X be irreducible, and let its defining prime ideal $I(X)$ be generated by m binomials

$$x_1^{a_{i1}} \dots x_l^{a_{il}} - \lambda_i x_1^{b_{i1}} \dots x_l^{b_{il}}, \quad 1 \leq i \leq m. \quad (*)$$

Consider the natural action of the torus $T_l = (K^*)^l$ on \mathbb{A}^l ,

$$(t_1, \dots, t_l) \cdot (a_1, \dots, a_l) = (t_1 a_1, \dots, t_l a_l).$$

Let $X(T_l) = \text{Hom}(T_l, \mathbb{G}_m)$ be the character group of T_l , and let $\varepsilon_i \in X(T_l)$ be the character

$$\varepsilon_i(t_1, \dots, t_l) = t_i, \quad 1 \leq i \leq l.$$

For $1 \leq i \leq m$, let

$$\varphi_i = \sum_{r=1}^l (a_{ir} - b_{ir}) \varepsilon_r.$$

Set $T = \bigcap_{i=1}^m \ker \varphi_i$, and $X^\circ = \{(x_1, \dots, x_l) \in X \mid x_i \neq 0, \text{ for all } 1 \leq i \leq l\}$.

Proposition 1.11. *Let notations be as above.*

- (1) *There is a canonical action of T on X .*
- (2) *X° is T -stable. Further, the action of T on X° is simple and transitive.*
- (3) *T is a subtorus of T_l , and X is an equivariant affine embedding of T .*

Proof. (1) We consider the (obvious) action of T on \mathbb{A}^l . Let $(x_1, \dots, x_l) \in X$, $\mathbf{t} = (t_1, \dots, t_l) \in T$, and $(y_1, \dots, y_l) = \mathbf{t} \cdot (x_1, \dots, x_l) = (t_1 x_1, \dots, t_l x_l)$. Using the fact that (x_1, \dots, x_l) satisfies $(*)$, we obtain

$$y_1^{a_{i1}} \dots y_l^{a_{il}} = t_1^{a_{i1}} \dots t_l^{a_{il}} x_1^{a_{i1}} \dots x_l^{a_{il}} = \lambda_i t_1^{b_{i1}} \dots t_l^{b_{il}} x_1^{b_{i1}} \dots x_l^{b_{il}} = \lambda_i y_1^{b_{i1}} \dots y_l^{b_{il}},$$

for all $1 \leq i \leq m$, i.e., $(y_1, \dots, y_l) \in X$. Hence $\mathbf{t} \cdot (a_1, \dots, a_l) \in X$ for all $\mathbf{t} \in T$.

(2) Let $\mathbf{x} = (x_1, \dots, x_l) \in X^\circ$, and $\mathbf{t} = (t_1, \dots, t_l) \in T$. Then, clearly $\mathbf{t} \cdot \mathbf{x} \in X^\circ$. Considering \mathbf{x} as a point in \mathbb{A}^l , the isotropy subgroup in T_l at \mathbf{x} is $\{\text{id}\}$. Hence the isotropy subgroup in T at x is also $\{\text{id}\}$. Thus the action of T on X° is simple.

Let $(x_1, \dots, x_l), (x'_1, \dots, x'_l) \in X^\circ$. Set $\mathbf{t} = (t_1, \dots, t_l)$, where $t_i = x_i/x'_i$. Then, clearly $\mathbf{t} \in T$. Thus $(x_1, \dots, x_l) = \mathbf{t} \cdot (x'_1, \dots, x'_l)$. Hence the action of T on X° is simple and transitive.

(3) Now, fixing a point $\mathbf{x} \in X^\circ$, we obtain from (2) that the orbit map $\mathbf{t} \mapsto \mathbf{t} \cdot \mathbf{x}$ is in fact an isomorphism of T onto X° . Also, since X is not contained in any of the coordinate hyperplanes, the open set $X_i = \{(x_1, \dots, x_l) \in X \mid x_i \neq 0\}$ is nonempty for all $1 \leq i \leq l$. The irreducibility of X implies that the sets X_i , $1 \leq i \leq l$, are open dense in X , and hence their intersection

$$X^\circ = \bigcap_{i=1}^l X_i = \{(x_1, \dots, x_l) \in X \mid x_i \neq 0 \text{ for any } i\}$$

is an open dense set in X , and thus X° is irreducible. This implies that T is irreducible (and hence connected). Thus T is a subtorus of T_l . The assertion that X is an equivariant affine embedding of T follows from (1) and (2). \square

Remark 1.12. By Proposition 1.7, we have that the ideal $I(X)$ is a toric ideal in the sense of Definition 1.4; more precisely, $I(X) = I_{\mathcal{A}}$, where $\mathcal{A} = \{\rho_i, 1 \leq i \leq l\}$, where $\rho_i = \varepsilon_i|_T$ (here $K[T]$ is identified with $K[t_1^{\pm 1}, \dots, t_d^{\pm 1}]$, and the character group $X(T)$ with \mathbb{Z}^d).

Summary: Summarizing we have established

- (i) bijections between $\{\text{equivariant affine toroidal embeddings}\}$, $\{\text{toric ideals}\}$ and $\{\text{binomial prime ideals}\}$.
- (ii) a bijection between $\{\text{affine toric varieties with torus action by } T\}$, $\{\text{finitely generated, saturated, sub semigroups of } M \text{ which generate } M\}$, and $\{\text{strongly convex rational polyhedral cones inside } N_{\mathbb{R}}\}$.

1.13. Orbit decomposition in affine toric varieties. We shall preserve the notation of §1.2. As in §1.2, let X be an affine toric variety with a torus action by T . Let $K[X] = K[S_\sigma]$. We shall denote X also by X_σ .

Let us first recall the definition of faces of a convex polyhedral cone:

Definition 1.14. A face τ of σ is a convex polyhedral sub cone of σ of the form $\tau = \sigma \cap u^\perp$ for some $u \in \sigma^\vee$, and is denoted $\tau < \sigma$.

We have that X_τ is a principal open subset of X_σ , namely,

$$X_\tau = (X_\sigma)_u$$

Each face τ determines a (closed) point P_τ in X_σ , namely, it is the point corresponding to the maximal ideal in $K[X](= K[S_\sigma])$ given by the kernel of $e_\tau : K[S_\sigma] \rightarrow K$, where

for $u \in S_\sigma$, we have

$$e_\tau(u) = \begin{cases} 1, & \text{if } u \in \tau^\perp \\ 0, & \text{otherwise} \end{cases}$$

Remark 1.15. As a point in \mathbb{A}^l , P_τ may be identified with the l -tuple with 1 at the i -th place if χ_i is in τ^\perp , and 0 otherwise (here, as in §1.6.1, χ_i denotes the weight of the T -weight vector y_i - the class of x_i in $K[X_\sigma]$).

Let O_τ denote the T -orbit in X_σ through P_τ . In the sequel, we shall refer to P_τ as the *center* of O_τ . We have the following orbit decomposition in X_σ :

$$\begin{aligned} X_\sigma &= \bigcup_{\theta \leq \sigma} O_\theta \\ \overline{O_\tau} &= \bigcup_{\theta \geq \tau} O_\theta \\ \dim \tau + \dim O_\tau &= \dim X_\sigma \end{aligned}$$

(here, by dimension of a cone τ , one means the vector space dimension of the span of τ).

See [11], [18] for details.

Thus $\tau \mapsto \overline{O_\tau}$ defines an order reversing bijection between $\{\text{faces of } \sigma\}$ and $\{T\text{-orbit closures in } X_\sigma\}$. In particular, we have the following two extreme cases:

1. If τ is the 0-face, then $P_\tau = (1, \dots, 1)$ (as a point in \mathbb{A}^l), and $O_\tau = T$, and is contained in $X_\theta, \forall \theta < \sigma$. It is a dense open orbit.
2. If $\tau = \sigma$, then P_τ (as a point in \mathbb{A}^l) is the l -tuple $(0, \dots, 0)$ (we may suppose that σ spans $N_\mathbb{R}$); further, $O_\tau = \{P_\tau\}$, and is the unique closed orbit.

For a face τ , let us denote by N_τ the sublattice of N generated by the lattice points of τ . Let $N(\tau) = N/N_\tau$, and $M(\tau)$, the \mathbb{Z} -dual of $N(\tau)$. For a face θ of σ such that θ contains τ as a face, set

$$\theta_\tau := (\theta + (N_\tau)_\mathbb{R}) / (N_\tau)_\mathbb{R}$$

(here, for a lattice L , $L_\mathbb{R} = L \otimes \mathbb{R}$). Then the collection $\{\theta_\tau, \sigma \geq \theta \geq \tau\}$ forms the set of faces of a cone in $N(\tau)_\mathbb{R}$; we shall denote it by σ_τ .

Lemma 1.16. *For a face $\tau < \sigma$, $\overline{O_\tau}$ gets identified with the toric variety $\text{Spec } K[S_{\sigma_\tau}]$. Further, $K[\overline{O_\tau}] = K[S_\sigma \cap \tau^\perp]$.*

Proof. The first assertion follows from the description of the orbit decomposition (and the definition of σ_τ). For the second assertion, we have

$$S_{\sigma_\tau} = \sigma_\tau^\vee \cap M(\tau) = \sigma^\vee \cap (\tau^\perp \cap M) = S_\sigma \cap \tau^\perp$$

(note that we have, $M(\tau) = \tau^\perp \cap M$). □

2. GENERALITIES ON FINITE DISTRIBUTIVE LATTICES

We shall now study a special class of toric varieties, namely, the toric varieties associated to distributive lattices. We shall first collect some definitions on finite partially ordered sets. A partially ordered set is also called a poset.

Definition 2.1. A finite poset P is called *bounded* if it has a unique maximal, and a unique minimal element, denoted $\hat{1}$ and $\hat{0}$ respectively.

Definition 2.2. A totally ordered subset C of a finite poset P is called a *chain*, and the number $\#C - 1$ is called the *length* of the chain.

Definition 2.3. A bounded poset P is said to be *graded* (or also *ranked*) if all maximal chains have the same length (note that $\hat{1}$ and $\hat{0}$ belong to any maximal chain).

Definition 2.4. Let P be a graded poset. The length of a maximal chain in P is called the *rank* of P .

Definition 2.5. Let P be a graded poset. For $\lambda, \mu \in P$ with $\lambda \geq \mu$, the graded poset $\{\tau \in P \mid \mu \leq \tau \leq \lambda\}$ is called the *interval from μ to λ* , and denoted by $[\mu, \lambda]$. The rank of $[\mu, \lambda]$ is denoted by $l_\mu(\lambda)$; if $\mu = \hat{0}$, then we denote $l_\mu(\lambda)$ by just $l(\lambda)$.

Remark 2.6. In the sequel, we refer to $l(\lambda)$ as *the level of λ in \mathcal{L}* .

Definition 2.7. Let P be a graded poset, and $\lambda, \mu \in P$, with $\lambda \geq \mu$. The ordered pair (λ, μ) is called a *cover* (and we also say that λ *covers* μ) if $l_\mu(\lambda) = 1$.

2.8. Generalities on distributive lattices.

Definition 2.9. A *lattice* is a partially ordered set (\mathcal{L}, \leq) such that, for every pair of elements $x, y \in \mathcal{L}$, there exist elements $x \vee y$ and $x \wedge y$, called the *join*, respectively the *meet* of x and y , defined by:

$$\begin{aligned} x \vee y &\geq x, \ x \vee y \geq y, \text{ and if } z \geq x \text{ and } z \geq y, \text{ then } z \geq x \vee y, \\ x \wedge y &\leq x, \ x \wedge y \leq y, \text{ and if } z \leq x \text{ and } z \leq y, \text{ then } z \leq x \wedge y. \end{aligned}$$

It is easy to check that the operations \vee and \wedge are commutative and associative.

Definition 2.10. An element $z \in \mathcal{L}$ is called the *zero* of \mathcal{L} , denoted by $\hat{0}$, if $z \leq x$ for all x in \mathcal{L} . An element $z \in \mathcal{L}$ is called the *one* of \mathcal{L} , denoted by $\hat{1}$, if $z \geq x$ for all x in \mathcal{L} .

Definition 2.11. Given a lattice \mathcal{L} , a subset $\mathcal{L}' \subset \mathcal{L}$ is called a *sublattice* of \mathcal{L} if $x, y \in \mathcal{L}'$ implies $x \wedge y \in \mathcal{L}'$, $x \vee y \in \mathcal{L}'$.

Definition 2.12. Two lattices \mathcal{L}_1 and \mathcal{L}_2 are *isomorphic* if there exists a bijection $\varphi : \mathcal{L}_1 \rightarrow \mathcal{L}_2$ such that, for all $x, y \in \mathcal{L}_1$,

$$\varphi(x \vee y) = \varphi(x) \vee \varphi(y) \text{ and } \varphi(x \wedge y) = \varphi(x) \wedge \varphi(y).$$

Definition 2.13. A lattice is called *distributive* if the following identities hold:

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) \quad (1)$$

$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z). \quad (2)$$

Example: The lattice of all subsets of the set $\{1, 2, \dots, n\}$ is a distributive lattice, denoted by $\mathcal{B}(n)$, and called the *Boolean algebra of rank n* .

Definition 2.14. An element z of a lattice \mathcal{L} is called *join-irreducible* (respectively *meet-irreducible*) if $z = x \vee y$ (respectively $z = x \wedge y$) implies $z = x$ or $z = y$. The set of join-irreducible (respectively meet-irreducible) elements of \mathcal{L} is denoted by $J_{\mathcal{L}}$ (respectively $M_{\mathcal{L}}$), or just by J (respectively M) if no confusion is possible.

Definition 2.15. The set $J_{\mathcal{L}} \cap M_{\mathcal{L}}$ of join and meet-irreducible elements is denoted by $JM_{\mathcal{L}}$, or just JM if no confusion is possible.

Definition 2.16. A subset I of a poset P is called an *ideal* of P if for all $x, y \in P$,

$$x \in I \text{ and } y \leq x \text{ imply } y \in I.$$

Theorem 2.17. (*Birkhoff*) Let \mathcal{L} be a distributive lattice with $\hat{0}$, and P the poset of its nonzero join-irreducible elements. Then \mathcal{L} is isomorphic to the lattice of finite ideals of P , by means of the lattice isomorphism

$$\alpha \mapsto I_{\alpha} = \{\tau \in P \mid \tau \leq \alpha\}, \quad \alpha \in \mathcal{L}.$$

Definition 2.18. A quadruple of the form $(\tau, \phi, \tau \vee \phi, \tau \wedge \phi)$, with $\tau, \phi \in \mathcal{L}$ non-comparable is called a *diamond*, and is denoted by $D(\tau, \phi, \tau \vee \phi, \tau \wedge \phi)$ or also just $D(\tau, \phi)$. The pair (τ, ϕ) (respectively $(\tau \vee \phi, \tau \wedge \phi)$) is called the *skew* (respectively *main*) diagonal of the diamond $D(\tau, \phi)$.

Denote

$$Q(\mathcal{L}) = \{(\tau, \phi), \tau, \phi \mid \tau, \phi \text{ noncomparable}\}$$

In the sequel, $Q(\mathcal{L})$ will also be denoted by just Q .

The following Lemma is easily checked.

Lemma 2.19. *With the notations as above, we have*

- (a) $J = \{\tau \in \mathcal{L} \mid \text{there exists at most one cover of the form } (\tau, \lambda)\}.$
- (b) $M = \{\tau \in \mathcal{L} \mid \text{there exists at most one cover of the form } (\lambda, \tau)\}.$

For $\alpha \in \mathcal{L}$, let I_{α} be the ideal corresponding to α under the isomorphism in Theorem 2.17.

Lemma 2.20. *Let (τ, λ) be a cover in \mathcal{L} . Then I_{τ} equals $I_{\lambda} \dot{\cup} \{\beta\}$ for some $\beta \in J_{\mathcal{L}}$.*

Proof. If $\tau \in J_{\mathcal{L}}$, then λ is the unique element covered by τ ; it is clear that in this case that $I_{\tau} = I_{\lambda} \cup \{\tau\}$.

Let then $\tau \notin J_{\mathcal{L}}$. Let λ' be another element covered by τ . Let $\phi = \lambda \wedge \lambda'$.

Claim: $(\lambda, \phi), (\lambda', \phi)$ are both covers.

We shall prove that (λ, ϕ) is a cover, the proof being similar for (λ', ϕ) . If possible, let us assume that there exists a ϕ' such that $\lambda > \phi' > \phi$. Then we have $\lambda' \leq \phi' \vee \lambda' \leq \tau$. The fact that (τ, λ') is a cover implies that

$$(*) \quad \phi' \vee \lambda' = \tau$$

(note that $\phi' \vee \lambda' \neq \lambda'$; for, $\phi' \vee \lambda' = \lambda'$ would imply that $\lambda' > \phi'$, and this in turn would imply that $\phi (= \lambda \wedge \lambda') \geq \phi'$ which would contradict the assumption that $\phi' > \phi$). Also, we have,

$$\phi = \lambda \wedge \lambda' \geq \phi' \wedge \lambda'$$

(since $\lambda > \phi'$), and

$$\phi' \wedge \lambda' \geq \phi$$

(since, both ϕ' and λ' are greater than ϕ). Hence we obtain

$$(**) \quad \phi' \wedge \lambda' = \phi$$

Now in view of $(*)$ and $(**)$, we obtain that for the pair (λ, ϕ') (with $\lambda > \phi'$), we have

$$\lambda \vee \lambda' = \phi' \vee \lambda', \quad \lambda \wedge \lambda' = \phi' \wedge \lambda'$$

But this is not possible, since \mathcal{L} is a distributive lattice. Hence our assumption is wrong and Claim follows.

The above Claim together with induction on $l(\tau)$ (the level of τ (cf. Remark 2.6)) implies that $I_{\lambda} = I_{\phi} \cup \{\beta\}$, for some $\beta \in J_{\mathcal{L}}$. We have, $\beta \notin I_{\lambda'}$ (note that $\beta \in I_{\lambda'}$ would imply $I_{\lambda} \subseteq I_{\lambda'}$ (since $\phi \leq \lambda'$) which is not true). We have that $I_{\lambda'} \cup \{\beta\}$ is an ideal; for, any $\gamma < \beta$ (and $\gamma \neq \beta$) is in fact $\leq \phi$ (since $I_{\phi} \cup \{\beta\} = I_{\lambda}$ is an ideal), and hence is in $I_{\lambda'}$. Further, $I_{\lambda'} \cup \{\beta\}$ is clearly contained in I_{τ} . Hence if θ is the element of \mathcal{L} corresponding to the ideal $I_{\lambda'} \cup \{\beta\}$ (under the bijection given by Theorem 2.17), then we have, $\lambda' < \theta \leq \tau$. Hence, we obtain that $\theta = \tau$ (since (τ, λ') is a cover).

Starting point of induction: Let τ be an element of least length among all θ 's such that θ covers an element θ' . Then the above reasoning (especially, the Claim) implies clearly that $\tau \in J_{\mathcal{L}}$, and therefore, $I_{\tau} = I_{\lambda} \cup \{\tau\}$. \square

Combining the above Lemma with the fact that I_1 equals $J_{\mathcal{L}}$, we have

Corollary 2.21. *Any maximal chain in (the ranked poset) \mathcal{L} has cardinality equal to $\#J_{\mathcal{L}}$.*

3. THE VARIETY $X_{\mathcal{L}}$

Consider the polynomial algebra $K[X_{\alpha}, \alpha \in \mathcal{L}]$; let $I_{\mathcal{L}}$ be the ideal generated by $\{X_{\alpha}X_{\beta} - X_{\alpha \vee \beta}X_{\alpha \wedge \beta}, \alpha, \beta \in \mathcal{L}\}$. Then one knows (cf.[16]) that $K[X_{\alpha}, \alpha \in \mathcal{L}]/I_{\mathcal{L}}$ is a normal domain; in particular, we have that $I_{\mathcal{L}}$ is a prime ideal. Let $X_{\mathcal{L}}$ be the affine variety of the zeroes in K^l of $I_{\mathcal{L}}$ (here, $l = \#\mathcal{L}$). Then $X_{\mathcal{L}}$ is a affine normal variety defined by binomials. Let T be as in Proposition 1.11. Then by that Proposition, we have

Theorem 3.1. *$X_{\mathcal{L}}$ is a (normal) toric variety for the action by T .*

We shall call $X_{\mathcal{L}}$ a *Hibi toric variety*. We shall now show that $\dim X_{\mathcal{L}} = \#J_{\mathcal{L}}$. We follow the notation in §1.10.

Let $\mathcal{I} = \{(\tau, \phi, \tau \vee \phi, \tau \wedge \phi) \mid (\tau, \phi) \in Q\}$, where

$$Q = \{(\tau, \phi) \mid \tau, \phi \in \mathcal{L} \text{ non-comparable}\}.$$

Let $T_l = (K^*)^l$, $\pi : X(T_l) \rightarrow X(T)$ be the canonical map, given by restriction, and for $\chi \in X(T_l)$, denote $\pi(\chi)$ by $\overline{\chi}$. Let us fix a \mathbb{Z} -basis $\{\chi_{\tau} \mid \tau \in \mathcal{L}\}$ for $X(T_l)$. For a diamond $D = (\tau, \phi, \tau \vee \phi, \tau \wedge \phi) \in \mathcal{I}$, let $\chi_D = \chi_{\tau \vee \phi} + \chi_{\tau \wedge \phi} - \chi_{\tau} - \chi_{\phi}$.

Lemma 3.2. *We have*

- (1) $X(T) \simeq X(T_l)/\ker \pi$.
- (2) $\ker \pi$ is generated by $\{\chi_D \mid D \in \mathcal{I}\}$.

Proof. The restriction map $\pi : X(T_l) \rightarrow X(T)$ is, in fact, surjective, since T is a subtorus of T_l . Now (1) follows from this. The assertion (2) follows from the definition of T (cf. Proposition 1.11). \square

Let $X(J_{\mathcal{L}})$ be the \mathbb{Z} -span of $\{\chi_{\theta}, \theta \in J_{\mathcal{L}}\}$. As an immediate consequence of the above Lemma, we have

Corollary 3.3. *π maps $X(J_{\mathcal{L}})$ isomorphically onto its image.*

Proof. The result follows since there does not exist a diamond contained completely in $J_{\mathcal{L}}$. \square

Lemma 3.4. *Let $\alpha \in \mathcal{L}$. Then $\overline{\chi}_{\alpha}$ is in the \mathbb{Z} -span of $\{\overline{\chi}_{\theta}, \theta \leq \alpha, \theta \in J_{\mathcal{L}}\}$. In particular, $\{\overline{\chi}_{\theta}, \theta \in J_{\mathcal{L}}\}$ generates $X(T)$ as a \mathbb{Z} -module.*

Proof. We shall prove the result by induction on $l(\alpha)$, level of α (cf. Remark 2.6). If $l(\alpha) = 0$, then $\alpha = \hat{0}$, and the result is clear. Let then $l(\alpha) \geq 1$; this implies in particular that there exist elements in \mathcal{L} covered by α . The result is clear if $\alpha \in J_{\mathcal{L}}$.

Let then $\alpha \notin J_{\mathcal{L}}$. This implies that there exist α_1, α_2 both of which are covered by α . Then α_1, α_2 are non-comparable (clearly). We have $\alpha_1 \vee \alpha_2 = \alpha$; let $\beta = \alpha_1 \wedge \alpha_2$. In view of Lemma 3.2(2), we have,

$$\overline{\chi}_\alpha = \overline{\chi}_{\alpha_1} + \overline{\chi}_{\alpha_2} - \overline{\chi}_\beta$$

By induction, each term on the R.H.S. is in the \mathbb{Z} -span of $\{\overline{\chi}_\theta, \theta \leq \alpha, \theta \in J_\mathcal{L}\}$, and the result follows. \square

Combining the above Lemma with Corollary 3.3, we obtain

Proposition 3.5. *The set $\{\overline{\chi}_\tau \mid \tau \in J_\mathcal{L}\}$ is a \mathbb{Z} -basis for $X(T)$.*

Now, since $\dim X_\mathcal{L} = \dim T$, we obtain

Theorem 3.6. *The dimension of $X_\mathcal{L}$ is equal to $\#J_\mathcal{L}$.*

Combining the above theorem with Lemma 2.20 and Corollary 2.21, we obtain

Corollary 3.7. (1) *The dimension of $X_\mathcal{L}$ is equal to the cardinality of the set of elements in a maximal chain in \mathcal{L} .*
 (2) *Fix any chain $\beta_1 < \dots < \beta_d$, d being $\#J_\mathcal{L}$. Let γ_{i+1} be the element of $J_\mathcal{L}$ corresponding to the cover (β_{i+1}, β_i) , $i \geq 1$; set $\gamma_1 = \beta_1$. Then $J_\mathcal{L} = \{\gamma_1, \dots, \gamma_d\}$*

Definition 3.8. For a finite distributive lattice \mathcal{L} , we call the cardinality of $J_\mathcal{L}$ the *dimension* of \mathcal{L} , and we denote it by $\dim \mathcal{L}$. If \mathcal{L}' is a sublattice of \mathcal{L} , then the *codimension* of \mathcal{L}' in \mathcal{L} is defined as $\dim \mathcal{L} - \dim \mathcal{L}'$.

Definition 3.9. A sublattice \mathcal{L}' of \mathcal{L} is called an *embedded sublattice* of \mathcal{L} if

$$\tau, \phi \in \mathcal{L}, \quad \tau \vee \phi, \tau \wedge \phi \in \mathcal{L}' \quad \Rightarrow \quad \tau, \phi \in \mathcal{L}'.$$

Given a sublattice \mathcal{L}' of \mathcal{L} , let us consider the variety $X_{\mathcal{L}'}$, and consider the canonical embedding $X_{\mathcal{L}'} \hookrightarrow \mathbb{A}(\mathcal{L}') \hookrightarrow \mathbb{A}(\mathcal{L})$ (here $\mathbb{A}(\mathcal{L}') = \mathbb{A}^{\#\mathcal{L}'}$, $\mathbb{A}(\mathcal{L}) = \mathbb{A}^{\#\mathcal{L}}$).

Proposition 3.10. *$X_{\mathcal{L}'}$ is a subvariety of $X_\mathcal{L}$ if and only if \mathcal{L}' is an embedded sublattice of \mathcal{L} .*

Proof. Under the embedding $X_{\mathcal{L}'} \hookrightarrow \mathbb{A}(\mathcal{L})$, $X_{\mathcal{L}'}$ can be identified with

$$\{(x_\tau)_{\tau \in \mathcal{L}} \in \mathbb{A}(\mathcal{L}) \mid x_\tau = 0 \text{ if } \tau \notin \mathcal{L}', \ x_\tau x_\phi = x_{\tau \vee \phi} x_{\tau \wedge \phi} \text{ for } \tau, \phi \in \mathcal{L}' \text{ noncomparable} \}.$$

Let η' be the center of the open dense orbit in $X_{\mathcal{L}'}$ (cf. Remark 1.15,(1)); note that $\eta'_\tau \neq 0$ if and only if $\tau \in \mathcal{L}'$. We have that $X_{\mathcal{L}'} \subset X_\mathcal{L}$ if and only if $\eta' \in X_\mathcal{L}$.

Assume that $\eta' \in X_\mathcal{L}$. Let τ, ϕ be two noncomparable elements of \mathcal{L} such that $\tau \vee \phi, \tau \wedge \phi$ are both in \mathcal{L}' . We have to show that $\tau, \phi \in \mathcal{L}'$. If possible, let $\tau \notin \mathcal{L}'$. This implies $\eta'_\tau = 0$. Hence either $\eta'_{\tau \vee \phi} = 0$, or $\eta'_{\tau \wedge \phi} = 0$, since $\eta' \in X_\mathcal{L}$. But this is not possible (note that $\tau \vee \phi, \tau \wedge \phi$ are in \mathcal{L}' , and hence $\eta'_{\tau \vee \phi}$ and $\eta'_{\tau \wedge \phi}$ are both nonzero).

Assume now that \mathcal{L}' is an embedded sublattice. We have to show that $\eta' \in X_\mathcal{L}$. Let τ, ϕ be two non-comparable elements of \mathcal{L} . The fact that \mathcal{L}' is an embedded

sublattice implies that $\{\tau, \phi\} \subset \mathcal{L}'$ if and only if $\{\tau \vee \phi, \tau \wedge \phi\} \subset \mathcal{L}'$; further, when $\tau, \phi, \tau \vee \phi, \tau \wedge \phi$ are in \mathcal{L}' , we have

$$\eta'_\tau \eta'_\phi = \eta'_{\tau \vee \phi} \eta'_{\tau \wedge \phi}$$

Thus η' satisfies the defining equations of $X_{\mathcal{L}}$, and hence $\eta' \in X_{\mathcal{L}}$. \square

4. CONE AND DUAL CONE OF $X_{\mathcal{L}}$:

In this section we shall determine the cone and the dual cone of $X_{\mathcal{L}}$, \mathcal{L} being a finite distributive lattice. As in the previous sections, denote the poset of join-irreducibles by $J_{\mathcal{L}}$; let $d = \# J_{\mathcal{L}}$. Denote $J_{\mathcal{L}} = \{\beta_1, \dots, \beta_d\}$. Let T be a d -dimensional torus. Identifying T with $(K^*)^d$, let $\{f_\beta, \beta \in J_{\mathcal{L}}\}$ denote the standard \mathbb{Z} -basis for $X(T)$, namely, for $u \in T, u = (u_\beta, \beta \in J_{\mathcal{L}})$, $f_\beta(u) = u_\beta$. Then $K[X(T)]$ may be identified with $K[u_{\beta_1}^{\pm 1}, \dots, u_{\beta_d}^{\pm 1}]$, the ring of Laurent polynomials. If $\chi \in X(T)$, say $\chi = \sum a_\beta f_\beta$, then as an element of $K[X(T)]$, χ will also be denoted by $\prod u_\beta^{a_\beta}$.

Denote by $\mathcal{I}(J_{\mathcal{L}})$ the poset of ideals of $J_{\mathcal{L}}$. For $A \in \mathcal{I}(J_{\mathcal{L}})$, set

$$f_A := \sum_{z \in A} f_z$$

Recall (cf. Theorem 2.17) the bijection $\mathcal{L} \rightarrow \mathcal{I}(J_{\mathcal{L}}), \theta \mapsto A_\theta := \{\tau \in J_{\mathcal{L}} \mid \tau \leq \theta\}$.

Lemma 4.1. $\{f_{A_\theta}, \theta \in J_{\mathcal{L}}\}$ is a \mathbb{Z} -basis for $X(T)$.

Proof. Take a total order on $J_{\mathcal{L}}$ extending the partial order. Then the matrix expressing $\{f_{A_\theta}, \theta \in J_{\mathcal{L}}\}$ in terms of the basis $\{f_\theta, \theta \in J_{\mathcal{L}}\}$ is easily seen to be triangular with diagonal entries equal to 1. The result follows from this. \square

Let $\mathcal{A} = \{f_A, A \in \mathcal{I}(J_{\mathcal{L}})\}$. Then as a consequence of the above Lemma, we obtain

Corollary 4.2. \mathcal{A} generates $X(T)$; in particular, $d_{\mathcal{A}} = d$, $d_{\mathcal{A}}$ being as in Proposition 1.5.

For $A \in \mathcal{I}(J_{\mathcal{L}})$, denote by \mathbf{m}_A the monomial:

$$\mathbf{m}_A := \prod_{\tau \in A} u_\tau$$

in $K[X(T)]$. If α is the element of \mathcal{L} such that $I_\alpha = A$ (cf. Theorem 2.17), then we shall denote \mathbf{m}_A also by \mathbf{m}_α . Consider the surjective algebra map

$$F : K[X_\alpha, \alpha \in \mathcal{L}] \rightarrow K[\mathbf{m}_A, A \in \mathcal{I}(J_{\mathcal{L}})] (\subset K[X(T)]), X_\alpha \mapsto \mathbf{m}_A, A = I_\alpha$$

Then with notation as in §1.3 and Definition 1.4, we have, $F = \widehat{\pi_{\mathcal{A}}}$, $\ker F = I_{\mathcal{A}}$. Further, Proposition 1.5 and Corollary 4.2 imply the following

Proposition 4.3. $\mathcal{V}(I_{\mathcal{A}}) = \text{Spec } K[\mathbf{m}_A, A \in \mathcal{I}(J_{\mathcal{L}})]$ and is of dimension $d (= \# J_{\mathcal{L}})$ (here, $\mathcal{V}(I_{\mathcal{A}})$ is as in Proposition 1.5)

Let us denote $\mathcal{V}(I_{\mathcal{A}})$ by Y .

Lemma 4.4. *Kernel of F is generated by $\{X_\alpha X_\beta - X_{\alpha \vee \beta} X_{\alpha \wedge \beta}, \alpha, \beta \in \mathcal{L}\}$*

Proof. We have (in view of Lemma 2.20) that if (α, λ) is a cover, then I_α equals $I_\lambda \dot{\cup} \{\beta\}$ for some $\beta \in J_\mathcal{L}$. A repeated application of this result implies that if $\gamma \leq \alpha$, and $m = l(\alpha) - l(\gamma)$ (here, $l(\beta)$ denotes the level of β (cf. Remark 2.6)), then there exist $\alpha_1 \cdots, \alpha_m$ in $J_\mathcal{L}$ such that $I_\alpha \setminus I_\gamma = \{\alpha_1 \cdots, \alpha_m\}$. Let now β, β' be two non-comparable elements in \mathcal{L} . Let $\alpha = \beta \vee \beta', \gamma = \beta \wedge \beta'$. Let $l(\beta) - l(\gamma) = r, l(\beta') - l(\gamma) = s$. Then there exist β_1, \dots, β_r , and $\beta'_1, \dots, \beta'_s$ in $J_\mathcal{L}$ such that

$$\begin{aligned} I_\beta \setminus I_\gamma &= \{\beta_1, \dots, \beta_r\} \\ I_{\beta'} \setminus I_\gamma &= \{\beta'_1, \dots, \beta'_s\} \\ I_\alpha \setminus I_\beta &= \{\beta'_1, \dots, \beta'_s\} \\ I_\alpha \setminus I_{\beta'} &= \{\beta_1, \dots, \beta_r\} \end{aligned}$$

Hence we obtain

$$\begin{aligned} \mathbf{m}_\beta &= \mathbf{m}_\gamma u_{\beta_1} \cdots u_{\beta_r} \\ \mathbf{m}_{\beta'} &= \mathbf{m}_\gamma u_{\beta'_1} \cdots u_{\beta'_s} \\ \mathbf{m}_\alpha &= \mathbf{m}_{\beta'} u_{\beta'_1} \cdots u_{\beta'_s} \\ \mathbf{m}_\alpha &= \mathbf{m}_{\beta'} u_{\beta_1} \cdots u_{\beta_r} \end{aligned}$$

From this it follows that

$$\mathbf{m}_\alpha \mathbf{m}_\gamma = \mathbf{m}_\beta \mathbf{m}_{\beta'}$$

Thus for each diamond in \mathcal{L} , i.e., a quadruple $(\beta, \beta', \beta \vee \beta', \beta \wedge \beta')$, we have $X_\beta X_{\beta'} - X_{\beta \vee \beta'} X_{\beta \wedge \beta'}$ is in the kernel of the surjective map F . Hence F factors through $K[X_\mathcal{L}]$; hence, we obtain closed immersions (of irreducible varieties):

$$Y \hookrightarrow X_\mathcal{L} \hookrightarrow \mathbb{A}^{\#\mathcal{L}}$$

(Y being $\mathcal{V}(I_\mathcal{A})$). But dimension considerations imply that $Y = X_\mathcal{L}$ (note that in view of Proposition 4.3, $\dim Y = d = \dim X_\mathcal{L}$ (cf. Theorem 3.6)), and the result follows. \square

As an immediate consequence, we obtain (as seen in the proof of the above Lemma)

Theorem 4.5. *We have an isomorphism $K[X_\mathcal{L}] \cong K[\mathbf{m}_A, A \in \mathcal{I}(J_\mathcal{L})]$.*

Denote $M := X(T)$, the character group of T . Let $N = \mathbb{Z}$ -dual of M , and $\{e_y, y \in J_\mathcal{L}\}$ be the basis of N dual to $\{f_z, z \in J_\mathcal{L}\}$. As above, for $A \in \mathcal{I}(J_\mathcal{L})$, let

$$f_A := \sum_{z \in A} f_z$$

Let $V = N_\mathbb{R} (= N \otimes_\mathbb{Z} \mathbb{R})$. Let $\sigma \subset V$ be the cone such that $X_\mathcal{L} = X_\sigma$. Let $\sigma^\vee \subset V^*$ be the cone dual to σ . Let $S_\sigma = \sigma^\vee \cap M$, so that $K[X_\mathcal{L}]$ equals the semi group algebra $K[S_\sigma]$.

As an immediate consequence of Theorem 4.5, we have

Proposition 4.6. *The semigroup S_σ is generated by $\{f_A, A \in \mathcal{I}(J_\mathcal{L})\}$.*

Let $M(J_\mathcal{L})$ be the set of maximal elements in the poset $J_\mathcal{L}$. Let $Z(J_\mathcal{L})$ denote the set of all covers in the poset $J_\mathcal{L}$ (i.e., $(z, z'), z > z'$ in the poset $J_\mathcal{L}$, and there is no other element $y \in J_\mathcal{L}$ such that $z > y > z'$). For a cover $(y, y') \in Z(J_\mathcal{L})$, denote

$$v_{y,y'} := e_{y'} - e_y$$

Proposition 4.7. *The cone σ is generated by $\{e_z, z \in M(J_\mathcal{L}), v_{y,y'}, (y, y') \in Z(J_\mathcal{L})\}$.*

Proof. Let θ be the cone generated by $\{e_z, z \in M(J_\mathcal{L}), v_{y,y'}, (y, y') \in Z(J_\mathcal{L})\}$. Then clearly any u in S_σ is non-negative on the generators of θ , and hence $\sigma^\vee \subseteq \theta^\vee$. We shall now show that $\theta^\vee \subseteq \sigma^\vee$, equivalently, we shall show that $S_\theta \subseteq S_\sigma$. Let $f \in M$, say, $f = \sum_{z \in J_\mathcal{L}} a_z f_z$. Then it is clear that f is in S_θ if and only if

$$(*) \quad a_z \geq 0, \text{ for } z \text{ maximal in } J_\mathcal{L}, \text{ and } a_x \geq a_y, \text{ for } x, y \in J_\mathcal{L}, x < y$$

Claim: Let $f \in M$. Then f has property (*) if and only if $f = \sum_{A \in \mathcal{I}(J_\mathcal{L})} c_A f_A$, $c_A \in \mathbb{Z}_+$.

Note that Claim implies that $S_\theta \subseteq S_\sigma$, and the required result follows.

Proof of Claim: The implication \Leftarrow is clear.

The implication \Rightarrow : Let $f \in S_\theta$, say, $f = \sum_{z \in J_\mathcal{L}} a_z f_z$. The hypothesis that f has property (*) implies that a_z being non-negative for z maximal in $J_\mathcal{L}$, a_x is non-negative for all $x \in J_\mathcal{L}$. Thus $f = \sum_{z \in J_\mathcal{L}} a_z f_z, a_z \in \mathbb{Z}_+, z \in J_\mathcal{L}$. Further, the property in (*) that $a_x \geq a_y$ if $x < y$ implies that

$$\{x \mid a_x \neq 0\}$$

is an ideal in $J_\mathcal{L}$. Call it A ; in the sequel we shall denote A also by I_f . Let $m = \min\{a_x, x \in A\}$. Then either $f = m f_A$ in which case the Claim follows, or $f = m f_A + f_1$, where f_1 is in S_θ (note that f_1 also has property (*)); further, I_{f_1} is a proper subset of A . Thus proceeding we arrive at positive integers m_1, \dots, m_r , elements f_1, \dots, f_r (in S_θ) with $A_i := I_{f_i}$, an ideal in $J_\mathcal{L}$, and a proper subset of $A_{i-1} := I_{f_{i-1}}$ such that

$$f = m f_A + m_1 f_{A_1} + \dots + m_r f_{A_r}$$

(In fact, at the last step, we have, $f_r = m_r f_{A_r}$.) The Claim and hence the required result now follows. \square

4.8. Analysis of faces of σ . We shall concern ourselves just with the closed points in $X_\mathcal{L}$. So in the sequel, by a point in $X_\mathcal{L}$, we shall mean a closed point. Let τ be a face of σ . Let P_τ be the distinguished point of O_τ with the associated maximal ideal

being the kernel of the map

$$K[S_\sigma] \rightarrow K,$$

$$u \in S_\sigma, u \mapsto \begin{cases} 1, & \text{if } u \in \tau^\perp \\ 0, & \text{otherwise} \end{cases}$$

Then for a point $P \in X_{\mathcal{L}}$ (identified with a point in \mathbb{A}^l), denoting by $P(\alpha)$, the α -th co-ordinate of P , we have,

$$P_\tau(\alpha) = \begin{cases} 1, & \text{if } f_{I_\alpha} \in \tau^\perp \\ 0, & \text{otherwise} \end{cases}$$

With notation as above, let

$$D_\tau = \{\alpha \in \mathcal{L} \mid P_\tau(\alpha) \neq 0\}$$

We have,

Lemma 4.9. *D_τ is an embedded sublattice of \mathcal{L} .*

Proof. Let θ, δ be a pair of non-comparable elements in D_τ . The fact that $P_\tau(\theta), P_\tau(\delta)$ are non-zero, together with the diamond relation $x_\theta x_\delta = x_{\theta \vee \delta} x_{\theta \wedge \delta}$ implies that $P_\tau(\theta \vee \delta), P_\tau(\theta \wedge \delta)$ are again non-zero, and are equal to 1 (note that any non-zero co-ordinate in P_τ is in fact equal to 1). Thus, D_τ is a sublattice of \mathcal{L} .

The above reasoning implies that if θ, δ in D_τ form the main diagonal in a diamond D in \mathcal{L} , then $P_\tau(\alpha), P_\tau(\beta)$ are non-zero, and are equal to 1, where α, β form the skew diagonal of D . Hence α, β are in D , and hence $D \subset D_\tau$. Thus we obtain that D_τ is an embedded sublattice of \mathcal{L} . \square

Conversely, we have

Lemma 4.10. *Let \mathcal{D} be an embedded sublattice in \mathcal{L} . Then \mathcal{D} determines a unique face τ of σ such that D_τ equals \mathcal{D} .*

Proof. Denote P to be the point in $\mathbb{A}^{\#\mathcal{L}}$ with

$$P(\alpha) = \begin{cases} 1, & \text{if } \alpha \in \mathcal{D} \\ 0, & \text{otherwise} \end{cases}$$

Then P is in \mathcal{L} (since \mathcal{D} is an embedded sublattice in \mathcal{L} , the co-ordinates of P satisfy all the diamond relations). Set

$$u = \sum_{\alpha \in \mathcal{D}} f_{I_\alpha}, \tau = \sigma \cap u^\perp$$

Then clearly, $P = P_\tau$ and $D_\tau = \mathcal{D}$ \square

Thus in view of the two Lemmas above, we have a bijection

$$\{\text{faces of } \sigma\} \xleftrightarrow{\text{bij}} \{\text{embedded sublattices of } \mathcal{L}\}$$

Proposition 4.11. *Let τ be a face of σ . Then we have $\overline{O_\tau} = X_{D_\tau}$.*

Proof. Recall (cf. Lemma 1.16) that $K[\overline{O_\tau}] = K[S_\sigma \cap \tau^\perp]$. From the description of P_τ , we have that τ^\perp is the span of $\{f_{I_\alpha}, \alpha \in D_\tau\}$; hence

$$(*) \quad S_\sigma \cap \tau^\perp = \left\{ \sum_{\alpha \in D_\tau} c_\alpha f_{I_\alpha}, c_\alpha \in \mathbb{Z}^+ \right\}$$

Thus we obtain

$$(**) \quad S_\sigma \cap \tau^\perp = S_{\sigma_\tau}$$

(here, σ_τ is as in Lemma 1.16). On the other hand, if η is the cone associated to the toric variety X_{D_τ} , then by Proposition 4.6 we have that S_η is the semigroup generated by $\{f_{I_\alpha}, \alpha \in D_\tau\}$. Hence we obtain that $\eta = \sigma_\tau$ (in view of $(*)$ and $(**)$). The required result now follows. \square

5. GENERATION OF THE COTANGENT SPACE AT P_τ

For $\alpha \in \mathcal{L}$, let us denote the image of X_α in $R_\mathcal{L}$ (under the surjective map $K[X_\alpha, \alpha \in \mathcal{L}] \rightarrow R_\mathcal{L}$) by x_α . Let $R_\tau = K[X(D_\tau)]$, the co-ordinate ring of $X(D_\tau)$, and $\pi_\tau : R_\mathcal{L} \rightarrow R_\tau$ be the canonical surjective map induced by the closed immersion $X(D_\tau) \hookrightarrow X_\mathcal{L}$; clearly, kernel of π_τ is generated by $\{x_\theta, \theta \in \mathcal{L} \setminus D_\tau\}$. Set

$$F_\alpha = \begin{cases} x_\alpha, & \text{if } \alpha \notin D_\tau \\ 1 - x_\alpha, & \text{if } \alpha \in D_\tau \end{cases}$$

Denoting by M_τ , the maximal ideal in $R_\mathcal{L}$ corresponding to P_τ , we have (cf. §4.8)

Lemma 5.1. *The ideal M_τ is generated by $\{F_\alpha, \alpha \in \mathcal{L}\}$.*

5.1.1. *A set of generators for the cotangent space M_τ/M_τ^2 .* For $F \in M_\tau$, let \overline{F} denote the class of F in M_τ/M_τ^2 .

Lemma 5.2. *Fix a maximal chain Γ in D_τ . For any $\beta \in D_\tau \setminus \Gamma$, we have that in M_τ/M_τ^2 , \overline{F}_β is in the span of $\{\overline{F}_\gamma, \gamma \in \Gamma\}$*

Proof. Fix a $\beta \in D_\tau \setminus \Gamma$; denote by $l_\tau(\beta)$, the level (cf. Remark 2.6) of β considered as an element of the distributive lattice D_τ . We shall prove the result by induction on $l_\tau(\beta)$. If $l_\tau(\beta) = 0$, then β coincides with the (unique) minimal element of D_τ , and there is nothing to prove. Let then $l_\tau(\beta) \geq 1$. Let β' in D_τ be such that β covers β' . Then $I_\beta \setminus I_{\beta'} = \{\theta\}$, for an unique $\theta \in J_{D_\tau}$ (cf. Lemma 2.20). Then there exists a unique cover (γ, γ') in Γ such that $I_\gamma \setminus I_{\gamma'} = \{\theta\}$ (cf. Corollary 3.7,(2)).

Claim: $F_\beta - F_{\beta'} \equiv F_\gamma - F_{\gamma'} \pmod{M_\tau}$.

First observe that the fact that θ belongs to I_β, I_γ and does not belong to $I_{\beta'}, I_{\gamma'}$ implies

$$(*) \quad \gamma' \not\leq \beta, \beta' \not\leq \gamma$$

We now divide the proof of the Claim into the following two cases.

Case 1: $\gamma' < \beta$.

This implies that γ' is in fact less than β' , and $\gamma < \beta$ (since, $I_\beta \setminus I_{\beta'} = I_\gamma \setminus I_{\gamma'} = \{\theta\}$, and $\theta \notin I_{\gamma'}$); this in turn implies that $\beta' \not\leq \gamma$ (for, otherwise, $\beta' \leq \gamma$ would imply $\gamma' < \beta' < \gamma$, not possible, since, (γ, γ') is a cover). Let D be the diamond having (γ, β') as the skew diagonal. Now γ' being less than both γ and β' , we get that $\gamma' \leq \beta' \wedge \gamma$, and in fact equals $\beta' \wedge \gamma$ (note that the fact that $\gamma' \leq \beta' \wedge \gamma < \gamma$ together with the fact that (γ, γ') is a cover in D_τ implies that $\gamma' = \beta' \wedge \gamma$). In a similar way, we obtain that $\beta = \beta' \vee \gamma$. Now the diamond relation $x_{\beta'}x_\gamma = x_\beta x_{\gamma'}$ implies the claim in this case (by definition of F_ξ 's).

Case 2: $\gamma' \not\leq \beta$.

This implies that $\gamma \not\leq \beta$.

If $\gamma > \beta$, then clearly

$$\gamma = \gamma' \vee \beta, \beta' = \gamma' \wedge \beta$$

Claim in this case follows as in Case 1.

Let then $\gamma \not\leq \beta$. Then we have that $(\gamma, \beta), (\gamma', \beta')$ are non-comparable pairs (note that $\gamma' \leq \beta'$ if and only if $\gamma' < \beta$). Denote $\delta = \gamma \wedge \beta, \delta' = \gamma' \wedge \beta'$. The facts that

$$I_\delta = I_\gamma \cap I_\beta, I_{\delta'} = I_{\gamma'} \cap I_{\beta'}, I_\gamma = I_{\gamma'} \cup \{\theta\}, I_\beta = I_{\beta'} \cup \{\theta\}$$

imply that

$$I_\delta = I_{\delta'} \cup \{\theta\}$$

Hence we obtain

$$\begin{aligned} I_{\gamma'} \cap I_\delta &= I_{\gamma'} \cap I_{\delta'} = I_{\delta'}; I_{\gamma'} \cup I_\delta = I_\gamma \\ I_{\beta'} \cap I_\delta &= I_{\beta'} \cap I_{\delta'} = I_{\delta'}; I_{\beta'} \cup I_\delta = I_\beta \end{aligned}$$

Thus we obtain that

$$\begin{aligned} \gamma' \wedge \delta &= \delta', \gamma' \vee \delta = \gamma \\ \beta' \wedge \delta &= \delta', \beta' \vee \delta = \beta \end{aligned}$$

Considering the diamonds with skew diagonals $(\gamma', \delta), (\beta', \delta)$ respectively, the diamond relations

$$x_{\gamma'}x_\delta = x_\gamma x_{\delta'}, x_{\beta'}x_\delta = x_\beta x_{\delta'}$$

imply that in M_τ/M_τ^2 , we have the following relations

$$\overline{F}_\gamma - \overline{F}_{\gamma'} = \overline{F}_\delta - \overline{F}_{\delta'}; \overline{F}_\beta - \overline{F}_{\beta'} = \overline{F}_\delta - \overline{F}_{\delta'}$$

Hence we obtain that $F_\beta - F_{\beta'} \equiv F_\gamma - F_{\gamma'} \pmod{M_\tau}$ as required.

This completes the proof of the Claim. Note that Claim implies the required result (by induction on $l_\tau(\beta)$). (Note that when $l_\tau(\beta) = 1$, then β' is the (unique) minimal element of D_τ , and hence $\beta' \in \Gamma$. The result in this case follows from the Claim). \square

Under the (surjective) map $\pi_\tau : R_{\mathcal{L}} \rightarrow R_\tau$, denote $\pi_\tau(M_\tau)$ by M'_τ ; then π_τ induces a surjection $\pi_\tau : M_\tau/M_\tau^2 \rightarrow M'_\tau/M_\tau'^2$.

Corollary 5.3. $\{\pi_\tau(\overline{F}_\gamma), \gamma \in \Gamma\}$ is a basis for $M'_\tau/M_\tau'^2$.

Proof. We have that $\dim M'_\tau/M_\tau'^2 \geq \dim X(D_\tau) = \#\Gamma$ (cf. Corollary 3.7(1)). On the other hand, the above Lemma implies that $\dim M'_\tau/M_\tau'^2 \leq \#\Gamma$, and the result follows. \square

Lemma 5.4. Let $\alpha \in \mathcal{L} \setminus D_\tau$ be such that there exists an element $\beta \in D_\tau$ and a diamond D in \mathcal{L} such that

(1) (α, β) is a diagonal (main or skew) in D .

(2) $D \cap D_\tau = \{\beta\}$.

Then $F_\alpha \in M_\tau^2$.

Proof. Let us denote the remaining vertices of the diamond by θ, δ . Writing the diamond relation $x_\alpha x_\beta = x_\theta x_\delta$ in terms of the F_ξ 's, we have,

$$x_\alpha x_\beta - x_\theta x_\delta = F_\alpha(1 - F_\beta) - F_\theta F_\delta$$

Hence in $R_{\mathcal{L}}$, we have,

$$F_\alpha = F_\alpha F_\beta + F_\theta F_\delta$$

The required result follows from this. \square

5.4.1. *The set E_τ .* Define

$$E_\tau := \{\alpha \in \mathcal{L}, \alpha \text{ as in Lemma 5.4}\}$$

In the sequel, we shall refer to an element $\alpha \in E_\tau$ as an E_τ -element.

5.4.2. *The equivalence relation:* For two distinct elements $\theta, \delta \in \mathcal{L} \setminus D_\tau$, we say θ is *equivalent to δ* (and denote it as $\theta \sim \delta$) if there exists a sequence $\theta = \theta_1, \dots, \theta_n = \delta$ in $\mathcal{L} \setminus D_\tau$ such that (θ_i, θ_{i+1}) forms one side of a diamond in \mathcal{L} whose other side is contained in D_τ . For $\theta \in \mathcal{L} \setminus D_\tau$, we shall denote by $[\theta]$, the set of all elements of $\mathcal{L} \setminus D_\tau$ equivalent to θ , if there exists such a diamond as above having θ as one vertex; if no such diamond exists, then $[\theta]$ will denote the singleton set $\{\theta\}$. Clearly, for all θ in a given equivalence class, $F_\theta \pmod{M_\tau^2}$ is the same (by consideration of diamond relations), and we shall denote it by $\overline{F}_{[\theta]}$ or also just \overline{F}_θ . Note also that in view of Lemma 5.4, $\overline{F}_\theta = 0$ in M_τ/M_τ^2 , if $[\theta] \cap E_\tau \neq \emptyset$. We shall refer to $[\theta]$ as a E_τ -class or a non E_τ -class according as $[\theta] \cap E_\tau$ is non-empty or empty.

The sublattice $\Lambda_\tau(\Gamma)$: Fix a chain Γ in D_τ . Let $\Lambda_\tau(\Gamma)$ denote the union of all the maximal chains in \mathcal{L} containing Γ . Note that α in \mathcal{L} is in $\Lambda_\tau(\Gamma)$ if and only if α is comparable to every $\gamma \in \Gamma$.

Lemma 5.5. $\Lambda_\tau(\Gamma)$ is a sublattice of \mathcal{L} .

Proof. Let (θ, δ) be a pair of non-comparable elements in $\Lambda_\tau(\Gamma)$. Then for any $\gamma \in \Gamma$, we have (by definition of $\Lambda_\tau(\Gamma)$) that either γ is less than both θ and δ or greater than both θ and δ ; in the former case, γ is less than both $\theta \vee \delta$ and $\theta \wedge \delta$, and in the latter case, γ is greater than both $\theta \vee \delta$ and $\theta \wedge \delta$. \square

Remark 5.6. $\Lambda_\tau(\Gamma)$ need not be an embedded sublattice:

Take \mathcal{L} to be the distributive lattice consisting of $\{(i, j), 1 \leq i, j \leq 4\}$ with the partial order $(a_1, a_2) \geq (b_1, b_2) \Leftrightarrow a_r \geq b_r, r = 1, 2$. Take

$$D_\tau := \{(i, j), 2 \leq i, j \leq 3\}, \Gamma := \{(2, 2), (3, 2), (3, 3)\}$$

Then

$$\Lambda_\tau(\Gamma) = \Gamma \cup I_1 \cup I_2$$

where $I_1 = \{(1, 1), (1, 2), (2, 1)\}$, $I_2 = \{(3, 4), (4, 3), (4, 4)\}$. Consider $x = (2, 4), y = (4, 2)$; then $x \vee y = (4, 4), x \wedge y = (2, 2)$. Now $x \vee y, x \wedge y$ are in $\Lambda_\tau(\Gamma)$, but x, y are not in $\Lambda_\tau(\Gamma)$. Thus $\Lambda_\tau(\Gamma)$ is not an embedded sublattice.

Proposition 5.7. *Let $\theta_0 \in \mathcal{L} \setminus D_\tau$. Then $\theta_0 \sim \mu$, for some $\mu \in \Lambda_\tau(\Gamma) \cup E_\tau$.*

Proof. Let us denote $\mathcal{L}'_\tau = \mathcal{L} \setminus D_\tau$. By induction, let us suppose that the result holds for all $\theta \in \mathcal{L}'_\tau, \theta > \theta_0$; we shall see that the proof for the case when θ_0 is a maximal element in \mathcal{L}'_τ (the starting point of induction) is included in the proof for a general θ_0 .

If $\theta_0 \in \Lambda_\tau(\Gamma)$, then there is nothing to prove. Let then $\theta_0 \notin \Lambda_\tau(\Gamma)$. Fix γ_1 minimal in Γ such that θ_0 and γ_1 are non-comparable. Let $\xi = \gamma_1 \wedge \theta_0, \theta_1 = \gamma_1 \vee \theta_0$. We divide the proof into the following two cases.

Case 1: γ_1 is the minimal element of Γ (note that γ_1 is the minimal element of D_τ also).

We have $\xi \in \Lambda_\tau(\Gamma)$ (since $\xi < \gamma_1$); further, $\xi \notin D_\tau$ (again, since $\xi < \gamma_1$, the minimal element of D_τ).

Subcase 1(a): Let θ_1 be in D_τ . Then considering the diamond with $(\theta_1, \gamma_1), (\theta_0, \xi)$ as opposite sides, we have, $\theta_0 \sim \xi$, and the result follows (note that $\xi \in \Lambda_\tau(\Gamma)$).

Subcase 1(b): Let $\theta_1 \notin D_\tau$. This implies $\theta_0 \in E_\tau$, and the result follows.

Case 2: γ_1 is not the minimal element of Γ .

Let $\gamma_0 \in \Gamma$ be such that γ_0 is covered by γ_1 in D_τ . Then $\xi \geq \gamma_0$ (since, both θ_0 and γ_1 are $> \gamma_0$).

Subcase 2(a): Let $\xi > \gamma_0$. Then the fact that (γ_1, γ_0) is a cover in D_τ together with the relation $\gamma_0 < \xi < \gamma_1$ implies that $\xi \in \Lambda_\tau(\Gamma) \setminus \Gamma$; in particular, $\xi \notin D_\tau$. Then as in Case 1, we obtain that $\theta_0 \in E_\tau$ if $\theta_1 \notin D_\tau$, and $\theta_0 \sim \xi$, if $\theta_1 \in D_\tau$; and the result follows (again note that $\xi \in \Lambda_\tau(\Gamma)$).

Subcase 2(b): Let $\xi = \gamma_0$. We first note that $\theta_1 \notin D_\tau$; for $\theta_1 \in D_\tau$ would imply that $\theta_0 \in D_\tau$ (since D_τ is an embedded sublattice (cf. Lemma 4.9)). Hence by induction

hypothesis, we obtain that

$$(*) \quad \theta_1 \sim \eta, \text{ for some } \eta \in \Lambda_\tau(\Gamma) \cup E_\tau$$

Also, considering the diamond with $(\theta_1, \theta_0), (\gamma_1, \gamma_0)$ as opposite sides, we have, $\theta_0 \sim \theta_1$. Hence, the result follows in view of (*).

Note that the above proof includes the proof of the starting point of induction, namely, θ_0 is a maximal element in \mathcal{L}'_τ . To make this more precise, let γ_1, ξ, θ_1 be as above. Proceeding as above, we obtain (by maximality of θ_0) that $\theta_1 \in D_\tau$. Hence Subcases 1(b) and 2(b) do not exist (since in these cases $\theta_1 \notin D_\tau$). In Subcases 1(a) and 2(a), we have $\xi \in \Lambda_\tau(\Gamma)$, and $\theta_0 \sim \xi$. The result now follows in this case. \square

Let $Y_\tau(\Gamma) = \Lambda_\tau(\Gamma) \setminus E_\tau$. Let us write

$$\begin{aligned} Y_\tau(\Gamma) &= \Gamma \dot{\cup} Z_\tau(\Gamma) \\ G_\tau(\Gamma) &= \{[\theta], \theta \in Z_\tau(\Gamma), [\theta] \text{ is a non } E_\tau\text{-class}\} \end{aligned}$$

Combining the above Proposition with Lemmas 5.2, 5.4, we obtain

Proposition 5.8. M_τ/M_τ^2 is generated by $\{\overline{F}_{[\theta]}, [\theta] \in G_\tau(\Gamma)\} \dot{\cup} \{\overline{F}_\gamma, \gamma \in \Gamma\}$.

6. A BASIS FOR THE COTANGENT SPACE AT P_τ

In this section, we shall show that $\{\overline{F}_{[\theta]}, [\theta] \in G_\tau(\Gamma)\} \dot{\cup} \{\overline{F}_\gamma, \gamma \in \Gamma\}$ is in fact a basis for M_τ/M_τ^2 . We first recall some basic facts on tangent cones and tangent spaces.

Let $X = \text{Spec } R \hookrightarrow \mathbb{A}^l$ be an affine variety; let S be the polynomial algebra $K[X_1, \dots, X_l]$. Let $P \in X$, and let M_P be the maximal ideal in $K[X]$ corresponding to P (we are concerned only with closed points of X). Let $A = \mathcal{O}_{X,P}$, the stalk at P ; denote the unique maximal ideal in A by $M (= M_P R_{M_P})$. Then $\text{Spec } gr(A, M)$, where $gr(A, M) = \bigoplus_{j \in \mathbb{Z}_+} M^j/M^{j+1}$ is the *tangent cone* to X at P , and is denoted $TC_P X$.

Note that

$$gr(R, M_P) = gr(A, M)$$

Tangent cone & tangent space at P : Let $I(X)$ be the vanishing ideal of X for the embedding $X = \text{Spec } R \hookrightarrow \mathbb{A}^l$. Expanding $F \in I(X)$ in terms of the local co-ordinates at P , we have the following:

- $T_P(X)$, the tangent space to X at P is the zero locus of the linear forms of F , for all $F \in I(X)$, i.e., the degree one part in the (polynomial) local expression for F .
- $TC_P(X)$, the tangent cone to X at P is the zero locus of the initial forms (i.e., form of smallest degree) of F , for all $F \in I(X)$.

In the sequel, we shall denote the initial form of F by $IN(F)$.

We collect below some well-known facts on singularities of X :

- Facts:**
1. $\dim T_P X \geq \dim X$ with equality if and only if X is smooth at P .
 2. X is smooth at P if and only if $gr(R, M_P)$ is a polynomial algebra.

6.1. Determination of the degree one part of $J(\tau)$: We now take $X = X_{\mathcal{L}}, P = P_{\tau}$; we shall denote $I = I(X_{\mathcal{L}}), M_P = M_{\tau}$. As above, for $F \in M_{\tau}$, let \overline{F} denote the class of F in M_{τ}/M_{τ}^2 . Let $J(\tau)$ be the kernel of the surjective map

$$f_{\tau} : K[X_{\theta}, \theta \in \mathcal{L}] \rightarrow \text{gr}(R, M_{\tau}), \quad X_{\theta} \mapsto \overline{F_{\theta}}$$

For $r \in \mathbb{N}$, let $f_{\tau}^{(r)}$ be the restriction of f to the degree r part of the polynomial algebra $K[X_{\theta}, \theta \in \mathcal{L}]$. We are interested in

$$f_{\tau}^{(1)} : \bigoplus_{\theta \in \mathcal{L}} KX_{\theta} \rightarrow M_{\tau}/M_{\tau}^2$$

We shall first describe a complement to the kernel of $f_{\tau}^{(1)}$, and then deduce a basis for M_{τ}/M_{τ}^2 (which will turn out to be the set $\{\overline{F_{[\theta]}}, [\theta] \in G_{\tau}(\Gamma)\} \cup \{\overline{F_{\gamma}}, \gamma \in \Gamma\}$).

Let V_{τ} be the span of $\{\overline{F_{\beta}}, \beta \in D_{\tau}\}$ (in M_{τ}/M_{τ}^2); by Lemmas 5.2, we have that V_{τ} is spanned by $\{\overline{F_{\gamma}}, \gamma \in \Gamma\}$. Let W_{τ} be the span of $\{\overline{F_{\theta}}, \theta \notin D_{\tau}\}$ (in M_{τ}/M_{τ}^2).

Lemma 6.2. $\{\overline{F_{\gamma}}, \gamma \in \Gamma\}$ is a basis for V_{τ} .

Proof. The surjective map $\pi_{\tau} : M_{\tau}/M_{\tau}^2 \rightarrow M'_{\tau}/M'^2_{\tau}$ induces a surjection $V_{\tau} \rightarrow M'_{\tau}/M'^2_{\tau}$, and the result follows from Corollary 5.3. \square

Lemma 6.3. The sum $V_{\tau} + W_{\tau}$ is direct.

Proof. Let $v \in V_{\tau}$; by Lemma 6.2, we can write $v = \sum_{\gamma \in \Gamma} a_{\gamma} \overline{F_{\gamma}}$. Now if $v \in W_{\tau}$, then under the map

$$\pi_{\tau} : M_{\tau}/M_{\tau}^2 \rightarrow M'_{\tau}/M'^2_{\tau}$$

we obtain that $\pi_{\tau}(v) = 0$ (since, $W_{\tau} \subseteq \ker \pi_{\tau}$). Hence we obtain that $\pi_{\tau}(\sum_{\gamma \in \Gamma} a_{\gamma} \overline{F_{\gamma}}) = 0$

in M'_{τ}/M'^2_{τ} ; this in turn implies that $a_{\gamma} = 0, \forall \gamma$ (cf. Corollary 5.3). Hence $v = 0$ and the required result follows. \square

Let us write $\bigoplus_{\theta \in \mathcal{L}} KX_{\theta} = A_{\tau} \oplus B_{\tau}$ where

$$A_{\tau} = \bigoplus_{\beta \in D_{\tau}} KX_{\beta}, \quad B_{\tau} = \bigoplus_{\theta \notin D_{\tau}} KX_{\theta}$$

Write $f_{\tau}^{(1)} = g_{\tau}^{(1)} + h_{\tau}^{(1)}$, where $g_{\tau}^{(1)}$ (respectively $h_{\tau}^{(1)}$) is the restriction of $f_{\tau}^{(1)}$ to A_{τ} (respectively B_{τ}). Note that we have surjections

$$g_{\tau}^{(1)} : A_{\tau} \rightarrow V_{\tau}, \quad h_{\tau}^{(1)} : B_{\tau} \rightarrow W_{\tau}$$

As a consequence of Lemma 6.3, we get the following

Corollary 6.4. $\ker f_{\tau}^{(1)} = \ker g_{\tau}^{(1)} \oplus \ker h_{\tau}^{(1)}$

Proof. The inclusion “ \supseteq ” is clear. Let then $v \in \ker f_{\tau}^{(1)}$; write $v = a + b$, where $a \in A_{\tau}, b \in B_{\tau}$. Then denoting $a' = f_{\tau}^{(1)}(a), b' = f_{\tau}^{(1)}(b)$, we have, $0 = a' + b'$. Also, we have that $a' \in V_{\tau}, b' \in W_{\tau}$. Hence in view of Lemma 6.3, we obtain $a' = 0 = b'$. This implies that $a \in \ker g_{\tau}^{(1)}, b \in \ker h_{\tau}^{(1)}$, and the result follows. \square

Lemma 6.5. *The span of $\{X_\gamma, \gamma \in \Gamma\}$ is a complement to the kernel of $g_\tau^{(1)}$.*

Proof. We have, $g_\tau^{(1)}(X_\beta) = \overline{F_\beta}$. By Lemma 6.2, we have that $\{\overline{F_\gamma}, \gamma \in \Gamma\}$ is a basis for V_τ . The result now follows. \square

Let $\{\xi_1, \dots, \xi_r\}$ be a complete set of representatives for $G_\tau(\Gamma)$.

Lemma 6.6. *The span of $\{X_{\xi_1}, \dots, X_{\xi_r}\}$ is a complement to the kernel of $h_\tau^{(1)}$.*

Proof. A typical element F of $I(X_\mathcal{L})$ such that $\text{IN}(F)$ is in B_τ is of the form

$$F = F_1 + F_2$$

where F_1 is a linear sum of diamond relations arising from diamonds having precisely one vertex in D_τ , and F_2 is a linear sum of diamond relations arising from diamonds having precisely one side in D_τ . Note that in a typical term in F_1 , the linear term is of the form $a_\alpha X_\alpha$, $a_\alpha \in K$ where $\alpha \in E_\tau$; similarly, in a typical term in F_2 , the linear term is of the form $b_{\theta\delta}(X_\theta - X_\delta)$, $\theta, \delta \notin D_\tau$, $\theta \sim \delta$, $b_{\theta\delta} \in K$. Hence the kernel of $h_\tau^{(1)}$ is generated by $\{X_\alpha, \alpha \in E_\tau\} \cup \{(X_\theta - X_\delta), \theta \sim \delta\}$. The required result now follows in view of Proposition 5.7. \square

Theorem 6.7. *$\{\overline{F}_{[\theta]}, [\theta] \in G_\tau(\Gamma)\} \cup \{\overline{F}_\gamma, \gamma \in \Gamma\}$ is a basis for M_τ/M_τ^2 .*

Proof. In view of Lemmas 6.5, 6.6 and Corollary 6.4, we obtain that

$\bigoplus_{\gamma \in \Gamma} KX_\gamma \oplus \bigoplus_{1 \leq i \leq r} KX_{\xi_i}$ is a complement to the kernel of the surjective map $f_\tau^{(1)} : \bigoplus_{\theta \in \mathcal{L}} KX_\theta \rightarrow M_\tau/M_\tau^2$. The result now follows. \square

Let $T_\tau X_\mathcal{L}$ denote the tangent space to $X_\mathcal{L}$ at P_τ . As an immediate consequence of the above Theorem, we have the following

Corollary 6.8. *$\dim T_\tau X_\mathcal{L} = \#G_\tau(\Gamma) + \#\Gamma$.*

In view of the above Corollary, we obtain that $X_\mathcal{L}$ is singular along O_τ if and only if $\#G_\tau(\Gamma) + \#\Gamma > \#\mathcal{L}$. Let

$$\mathcal{S}_\mathcal{L} = \{\tau < \sigma \mid \#G_\tau(\Gamma) + \#\Gamma > \#\mathcal{L}\}$$

(here, σ is the cone associated to $X_\mathcal{L}$). We obtain from Proposition 4.11 the following

Theorem 6.9. *$\text{Sing } X_\mathcal{L} = \bigcup_{\tau \in \mathcal{S}_\mathcal{L}} X(D_\tau)$.*

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